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Black hole entropy in M-theory

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Abstract: Extremal black holes in M-theory compactification on $M \times S^1$ are microscopically represented by fivebranes wrapping $P \times S^1$, where M is a Calabi-Yau threefold and P is a four-cycle in M. Additional spacetime charges arise from momentum around the S^1 and expectation values for the self-dual three-form field strength in the fivebrane. The microscopic entropy of the fivebrane as a function of all the charges is determined from a two-dimensional $(0, 4)$ sigma model whose target space includes the fivebrane moduli space. This entropy is compared to the macroscopic formula. Precise agreement is found for both the tree-level and one-loop expressions.

Keywords: M- and F-theories and Other Generalizations, Branes in String Theory, Black Holes in String Theory.

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Contents

1 Introduction

Thanks to recent advances in string theory, the thermodynamic properties of black holes can be microscopically derived in a variety of contexts with remarkable precision [1]. This has led to new insights into the quantum structure of black holes as well as string theory itself. The macroscopic-microscopic correspondence has developed to the extent that semiclassical black holes have become a powerful tool for analyzing the quantum theories which describe their dynamics [2, 3, 4, 5].

In this paper we will compute and compare the macroscopic and microscopic entropies of extremal black holes arising in an M-theory compactification to four dimensions on $M \times S^1$, where M is a Calabi-Yau threefold, so that the unbroken spacetime supersymmetry is precisely $N = 2$. Some special cases were considered in [6] and a heuristic explanation for the more general case was attempted in [7, 8].

Several new features arise in the computation. The basic microscopic object which enters is the still-mysterious M-theory fivebrane, aspects of whose world-volume field theory enter the analysis. The resulting geometric picture is quite interesting, since it maps the black hole degrees of freedom to the different ways of deforming a "foam" of fivebranes. We are able to go beyond the tree-level Bekenstein-Hawking entropy and successfully compute and compare the one-loop corrections. The macroscopic correction at one loop is related to the entropy arising from entanglement of the quantum state inside and outside the horizon, while at the microscopic level it is a subleading contribution to the central charge of a two-dimensional conformal field theory. A curious feature of the analysis is that we never need to use string theory itself: everything follows from the properties of M-theory fivebranes in curved spaces. Indeed with 20/20 hindsight, this paper might have been written years ago following the discovery of eleven-dimensional supergravity, Calabi-Yau compactification, and the fivebrane as a spacetime soliton.

This paper is organized as follows. In section 2.1 we review the macroscopic entropy formula, and the explicit solution for general nonzero S^1 momentum, twobrane and fivebrane charges. In section 2.2 we compute the leading one loop correction which arises from the R^4 term in eleven-dimensional M-theory. In section 3 we describe the $(0,4)$ sigma model that counts the BPS states. In 3.1 we compute the central charge in terms of the homology class of the four cycle P. In 3.2 we relate antisymmetric tensor expectation values to twobrane charges and show that this affects the entropy via a shift in the effective S^1 momentum. In section 4 we briefly raise - but do not resolve - the issue of α' corrections in the string theory regime.

2 Macroscopic entropy

2.1 The area formula

In this subsection, we review the semiclassical area-entropy formula for an $N=2,\ d=4$ extremal black hole characterized by magnetic and electric charges (p^Λ, q_Λ). The asymptotic values of the vector moduli $Z^\Lambda = X^\Lambda/X^0,\ \Lambda = 0,1,\dots,n_V$, in the black hole solution are arbitrary. These moduli couple to the electromagnetic fields and accordingly vary as a function of the radius. At the horizon they approach a fixed point whose location in the moduli space depends only on the charges [9, 10]. The locations of these fixed points can be found by looking for supersymmetric solutions with constant moduli. The general formula is [11, 9][1]

$$p^\Lambda = Re[CX^\Lambda]\,, \tag{2.1}$$

$$q_\Lambda = Re[CF_\Lambda]\,, \tag{2.2}$$

where $F_\Lambda = \partial_L F$ are the holomorphic periods. The $2n_v+2$ real equations (2.1) and (2.2) determine the n_v+2 complex quantities (C, X^Λ) up to Kahler transformations. The Kahler potential K is given by

$$X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda = ie^{-K}\,. \tag{2.3}$$

Given the horizon values of the moduli determined by (2.1) and (2.2) the Bekenstein-Hawking entropy is

$$S_{BH} = \frac{1}{4}\text{Area} = \frac{\pi i C\bar{C}}{4}(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda)\,. \tag{2.4}$$

[1] We have redefined $C,\ X$ and F by factors of $e^{K/2}$ relative to [11] so that (X^Λ, F_Λ) is a holomorphic section.

For M-theory compactified on $M \times S^1$, where M is a Calabi-Yau threefold, the prepotential is

$$F(X) = D_{ABC}\frac{X^A X^B X^C}{X^0}\,, \tag{2.5}$$

where $A, B = 1, ..n_V$. The intersection numbers $6D_{ABC}$ are

$$6D_{ABC} \equiv \int_M \alpha_A \wedge \alpha_B \wedge \alpha_C\,, \tag{2.6}$$

where the α_A are an integral basis for $H^2(M; \mathbf{Z})$.

The fixed point equations (2.1), (2.2) have been solved for a general prepotential of the form (2.5) with the restriction $p_0 = 0$ [12, 13]. The fixed points are given by

$$\begin{aligned} CX^0 &= i\sqrt{\frac{D}{\hat{q}_0}}\,, \\ CX^A &= p^A + \frac{i}{6}\sqrt{\frac{D}{\hat{q}_0}} D^{AB} q_B\,, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} D &\equiv D_{ABC}p^A p^B p^C\,, \\ \hat{q}_0 &\equiv q_0 + \frac{1}{12}D^{AB}q_A q_B\,, \\ D_{AB} &\equiv D_{ABC}p^C\,, \\ D^{AB}D_{BC} &= \delta^A{}_C\,. \end{aligned} \tag{2.8}$$

The entropy then follows from (2.4) and (2.5) as

$$S = 2\pi\sqrt{D\hat{q}_0}\,. \tag{2.9}$$

In order that the long wavelength approximation to M-theory can be trusted, the volume V_M of M as well as the radius R of the S^1 should both be large in eleven-dimensional Planck units. V_M is a hypermultiplet scalar which is a freely adjustable constant throughout the black hole solution. $R^3 V_M$ is a scalar in a vector multiplet whose value at the horizon is proportional to $\sqrt{\hat{q}_0^3/D}$. Hence the validity of the semiclassical M-theory computation requires $\hat{q}_0^3 \gg D$. The validity of the long wavelength approximation requires not just that the total volume of M should be large at the horizon, but that the volume of any two-cycle in M should be large there. It follows from (2.7) that the Kahler class at the horizon is proportional to $P \equiv p^A\alpha_A$. Hence we require that P lies inside the Kahler cone, a restriction that will be extensively used later.

The above conditions were necessary to ensure that we can derive the effective four dimensional theory by simple Kaluza-Klein reduction. Demanding that the black hole supergravity solution is weakly curved further requires

$$D \gg V_M\,. \tag{2.10}$$

2.2 Macroscopic loop corrections

The macroscopic entropy (2.9) was derived from the classical leading low-energy effective $d = 4$, $N = 2$ supergravity action. This action has corrections from higher-dimension operators. Corrections to the underlying eleven-dimensional action are a power series in the Planck length while corrections to the four dimensional action also involve R and V_M. In general there is no known systematic procedure for computing these corrections in the M-theory regime of large R and large V_M. However, the correction we are after arises from a special term whose coefficient can be determined. This is the R^2 correction

$$S_1 = \frac{1}{96\pi} \int c_{2A} \mathrm{Im} Z^A R \wedge *R \,, \tag{2.11}$$

where

$$Z^A(x) = \frac{X^A}{X^0} \,, \tag{2.12}$$

are moduli fields and

$$c_{2A} \equiv \int_M c_2(TM) \wedge \alpha_A \,, \tag{2.13}$$

with c_2 the second Chern class of M. This term has a topological origin [14, 15, 16] and descends from the R^4 term in $d = 11$ [16]. In string perturbation theory – that is, in an expansion in $1/V_M$ – this R^4 term has been computed as a one loop correction [17]. It cannot be renormalized at higher loops because it is related to anomaly cancellation [18, 19].

In order to determine the effect of (2.11) on the entropy we first consider the special case that the moduli fields take their constant fixed point values throughout the black hole solution. It then follows from (2.7) that $\mathrm{Im} Z^A = -p^A \sqrt{\hat{q}^0/D}$. Using the fact that the Euclidean black hole solution is $R^2 \times S^2$ and has Euler character 2, (2.11) is then

$$S_1 = -c_{2A} p^A \frac{\pi}{6} \sqrt{\frac{\hat{q}^0}{D}} \,. \tag{2.14}$$

This correction to the effective action is a correction to $-\log Z = \beta F$, so the correction to the entropy will be $\delta S = \delta \log Z - \beta \partial_\beta \delta \log Z$. Since the correction (2.11) is topological (proportional to the Euler character), it will be independent of the temperature. The correction to the entropy is then

$$\Delta S = -S_1 = c_{2A} p^A \frac{\pi}{6} \sqrt{\frac{\hat{q}^0}{D}} \,. \tag{2.15}$$

Note that unlike the leading term (2.9), this term is of order zero, rather than quadratic in the charges. It will, however, be large in the case we are considering where $q^0 \gg D^{1/3}$.

In general, the value of Z^A at infinity is arbitrary, and Z^A will vary over the black hole solution. The microscopic entropy is a function only of the charges and is independent of the asymptotic values of the the moduli, because the number of BPS states cannot vary smoothly with the moduli. Hence the macroscopic entropy should also be independent of the moduli. The macroscopic origin of this in the spacetime solutions has been

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understood for tree level entropy in terms of fixed points as described above. We do not understand the macroscopic mechanism for moduli-independence of the one loop corrections for the general case when the moduli fields are not taken to be at their fixed points. This will require a detailed knowledge of the one loop corrections and supersymmetry transformation laws.

The one-loop correction to the effective action of course involves more terms than just (2.11). Some of these involve gauge fields which are nonzero for the black hole solution, and in principle might further correct the entropy. However the topological nature of (2.11) suggests that it should play a dominant role, and we shall indeed find that the induced correction to the entropy agrees with the microscopic prediction.[2]

3 Microscopic entropy

At the microscopic level, configurations with charges $(0, p^A, q_0, q_A)$ are obtained from a fivebrane wrapping the five-cycle $P \times S^1$ and carrying total momentum q_0 about the S^1. Here P is a four-cycle in M. Its cohomology class $[P] \in H^2(M, \mathbf{Z})$ can be expanded as $[P] = \sum_A p^A \Sigma_A$ where the p^A are charges and Σ_A are a basis of $H^2(M, \mathbf{Z})$; one can consider the Σ_A to be the cohomology classes of a basis of four-cycles in M. Nonzero q_A charge arises, as we discuss later, from exciting the self-dual antisymmetric tensor field on the fivebrane. Nonzero p^0 - not considered here - would arise from a Kaluza-Klein monopole on the S^1.

Supersymmetry requires that P should be holomorphic; it is of complex codimension one in M. We will need to work out the low energy effective field theory obtained by wrapping a fivebrane on $P \times S^1$. As explained at the end of section 2.1, $[P]$ is proportional to the effective Kahler class of M near the black hole horizon, and hence long-wavelength M-theory is a good description only if $[P]$ is inside the Kahler cone in $H^2(M, \mathbf{Z})$, so that M is smooth at the horizon, and moreover is large. These conditions mean that P is a "very ample divisor" in the language of algebraic geometry, and lead to many simplifications. One important simplification is that we can assume that P is smooth; in the moduli space of P's, one will encounter singularities at special points, but the generic P of given cohomology class is smooth. The fact that P is generically smooth means that we do not need to understand the behavior of M-theory with coincident or intersecting fivebranes; everywhere there is locally a single, isolated, smooth fivebrane.

Another important consequence of the fact that P is very ample is that there is a general method to describe the moduli of P. Given any complex divisor P in a complex manifold M, there is a holomorphic line bundle $\mathcal{L}$ and a holomorphic section s of $\mathcal{L}$ such that s vanishes precisely along P. Existence of such an s means that $c_1(\mathcal{L}) = [P]$. Conversely, any holomorphic section s' of $\mathcal{L}$ vanishes along a divisor P' whose cohomology class equals that of P. s' is uniquely determined by P' up to scaling by a complex constant ($s' \to \lambda s'$ with $\lambda \in \mathbf{C}^*$). So the moduli space $\mathcal{M}$ of divisors that are cohomologous to P is

[2] There are also subleading logarithmic corrections [20] not considered here.

a complex projective space that is obtained by projectivizing the vector space $H^0(M,\mathcal{L})$.

This statement holds for any divisor P in a complex manifold. For a general P, there would be no nice formula for the dimension of $H^0(M,\mathcal{L})$. There is, however, a nice index or Riemann-Roch formula for the alternating sum $w = \sum_{i=0}^{\dim M}(-1)^i \dim H^i(M,\mathcal{L})$. In fact,

$$w = \int_M e^P \,\mathrm{Td}(M)\,. \tag{3.1}$$

Here $\mathrm{Td}(M)$ is the Todd class of M. Also, we have used the fact that $c_1(\mathcal{L}) = [P]$, and we henceforth sometimes write simply P instead of $[P]$ for ease of reading the formulas. For a divisor in a Calabi-Yau threefold, one has $\mathrm{Td}(M) = 1 + c_2(M)/12$, with $c_2(M)$ the second Chern class. So the index formula can be evaluated to give

$$w = \int_M \left(\frac{P^3}{6} + \frac{1}{12} P c_2(M)\right)\,. \tag{3.2}$$

In (3.2), there is no requirement of ampleness. But for P very ample, $\dim H^i(M,\mathcal{L}) = 0$ when $i > 0$. So in this case, the definition of w reduces to $w = \dim H^0(M,\mathcal{L})$. The quantity w in (3.2) is accordingly for very ample P the number of complex parameters required to determine a holomorphic section s of $\mathcal{L}$. Because of the equivalence under $s \to \lambda s$, the number of complex moduli of P is $w-1$, so the number of real moduli of P is $2w-2$ or

$$d_p = \int_M (\frac{1}{3}P^3 + \frac{1}{6}P c_2(M)) - 2\,. \tag{3.3}$$

These moduli can vary as we move along the S^1. We take the radius R of the S^1 much bigger than the typical size of the Calabi Yau space, in the sense that $R^6 \gg V_M$. The low energy dynamics is then described by a two-dimensional sigma model on the S^1 as in [21]. This sigma model has $(0,4)$ supersymmetry. The chirality of the sigma model descends from the $(0,2)$ chirality of the field theory on the fivebrane. This sigma model contains the moduli of the four-cycle, as well as scalar fields corresponding to expectation values of the two index antisymmetric tensor potential that lives on the fivebrane world volume. As in [21], the microscopic entropy is given by the logarithm of the number of left-moving excitations with total momentum q_0. For large q_0 ($q_0 \gg c_L$), this asymptotically approaches

$$S_{micro} = 2\pi\sqrt{\frac{c_L \tilde{q}_0}{6}}\,, \tag{3.4}$$

where c_L is the left-moving (non-supersymmetric) central charge and $\tilde{q}_0$ is the momentum available to freely distribute among the left-moving oscillators.

The above discussion makes sense when the fivebranes are far apart from each other so that we can neglect their gravitational effects, and also when they are embedded in a large space where we keep just the low energy description of the fivebrane. This requires that $D \ll V_M$, which is just the opposite of the condition (2.10) for the validity of the supergravity analysis. Since V_M is in a hyper multiplet, we can presumably interpolate between these two regimes without changing the number of BPS states. In order to reduce

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the dynamics to the sigma model we needed that $R^6 \gg V_M$, which will require changing the value of a vector multiplet. We are assuming throughout that there are no jumping phenomena (or that they can be neglected) when we perform this change.

3.1 Computing c_L

In this section we compute the microscopic entropy for the case of large q_0, so that we may approximate $q_0 = \tilde{q}_0 = \hat{q}_0$. The shift in q_0 will be considered in the next section. The results will reproduce for this case the leading entropy (2.9) as well as the one-loop semiclassical correction (2.15).

One contribution to c_L comes from massless fields that arise from fluctuations in the moduli of P. Such fields propagate as left- and right-movers on the S^1, and the left-movers contribute to c_L. There are no fermions contributing to c_L for the following reason. Fermions arise from $(0, i)$ forms on P. They are right-moving for i even and left-moving for i odd. Hence the number of left-moving fermions is $b_1(P)$, the first Betti number of P. For P a very ample divisor in a Kahler manifold M, the Lefschetz hyperplane theorem says that $b_1(P) = b_1(M)$. For M a complex threefold whose holonomy is $SU(3)$ (and not a subgroup), $b_1(M) = 0$, and so there are no left-moving fermions.[3]

The other bosonic contribution to c_L comes from the the rank two antisymmetric tensor potential b that propagates on the fivebrane world-volume. To determine its dimensional reduction to S^1, it is important that the three-form field strength $h = db$ is self-dual. The reduction of b to S^1 gives b_2^+ right-moving massless scalars on S^1 and b_2^- left-moving ones, where b_2^+ and b_2^- are the dimensions of the space of self-dual and anti-self-dual two-forms on P. (The reduction of b does not give two-dimensional gauge fields, since the first Betti number $b_1(P)$ vanishes for a reason given in the next footnote.)

The Euler characteristic χ and signature σ of P can be expressed in terms of $b_2^\pm$ by $\sigma = b_2^+ - b_2^-$ and $\chi = 2 + b_2^+ + b_2^-$.[4] χ and σ can be computed as follows. It suffices to know the Chern classes $c_1(P)$ and $c_2(P)$, since for a two-dimensional complex manifold P one has

$$\begin{aligned} \chi &= \int_P c_2(P) \,, \\ \sigma &= -\tfrac{2}{3}\chi + \tfrac{1}{3} \textstyle\int_P c_1(P)^2 \,. \end{aligned} \tag{3.5}$$

The Chern classes of P can be computed in terms of the Chern classes of M and the cohomology class $[P]$. Let TP and TM be the tangent bundles to P and M, so that by definition $c_i(P) = c_i(TP)$, $c_i(M) = c_i(TM)$. Let $TM|_P$, $\mathcal{L}|_P$ denote the restrictions of

[3] If the holonomy of M is a proper subgroup of $SU(3)$ – in other words, if M is a six-torus or a two-torus times K3 – then $b_1(M) \neq 0$, and there would be left moving fermions. For such M, the unbroken supersymmetry is greater than the $N = 2$ assumed in the present paper, and a few other details of the exposition would also be modified.

[4] In general, for a four-manifold P, $\chi(P) = 2 - 2b_1(P) + b_2^+ + b_2^-$, where $b_1(P)$ is the first Betti number of P, but we assume here that M is a complex threefold whose holonomy is $SU(3)$ (and not a subgroup). In this case, as discussed above, $b_1(M) = b_1(P) = 0$, so the formula for the Euler characteristic of P reduces to $\chi = 2 + b_2^+ + b_2^-$.

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TM and $\mathcal{L}$ to P. There is an exact sequence

$$0 \to TP \to TM|_P \to \mathcal{L}|_P \to 0\,, \tag{3.6}$$

which expresses the fact that the restriction of $\mathcal{L}$ to P can be understood as the normal bundle to P in M.[5] Hence, if $c = 1 + c_1 + c_2 + \ldots$ is the total Chern class, we have

$$c(TM|_P) = c(TP)c(\mathcal{L}|_P)\,. \tag{3.7}$$

Because M is Calabi-Yau, we have $c_1(TM) = 0$, and hence $c(TM|_P) = 1 + c_2(M) + \ldots$. So (3.7) gives the relations $c_1(P) = -c_1(\mathcal{L}) = -[P]$, and $c_2(P) = c_2(M) + [P]^2$.[6] With these relations, we can use (3.5) to express χ and σ in terms of $\int_P c_2(M)$ and $\int_P [P]^2$.

It is now convenient to note that if A is any cohomology class on M, whose restriction to P we write also as A, then $\int_P A = \int_M P \cdot A$. Using this principle, we can write $\chi(P)$ and $\sigma(P)$ in terms of integrals on M. Writing simply P instead of $[P]$, we get

$$\begin{aligned} \chi &= \int_P c_2(P) \\ &= \int_P (c_2(M) + c_1^2(P)) \\ &= \int_M (Pc_2(M) + P^3)\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \sigma &= -\tfrac{2}{3}\chi + \tfrac{1}{3}\textstyle\int_P c_1(P)^2 \\ &= -\textstyle\int_M (\tfrac{1}{3}P^3 + \tfrac{2}{3}Pc_2(TM))\,. \end{aligned} \tag{3.9}$$

Expressing $b_2^\pm$ in terms of χ and σ by $b_2^\pm = \frac{1}{2}(\chi \pm \sigma) - 1$ yields

$$b_2^- = \int_M \left(\frac{2}{3}P^3 + \frac{5}{6}Pc_2(TM)\right) - 1\,, \tag{3.10}$$

$$b_2^+ = \int_M (\frac{1}{3}P^3 + \frac{1}{6}Pc_2(TM)) - 1\,. \tag{3.11}$$

The total number of left- and right-moving massless bosons is $N_L^B = d_p + b_2^- + 3$, $N_R^B = d_p + b_2^+ + 3$, where in each case $+3$ is the contribution of three translational zero modes. So in terms of

$$c_2 \cdot P = \int_M Pc_2(TM)\,, \tag{3.12}$$

[5]For instance, if s is the section of $\mathcal{L}$ that vanishes along P, and s' is any other section, then the vanishing of $s + \epsilon s'$ defines a divisor P_ϵ that is homologous to P. To first order in ϵ, the ϵ-dependence of P_ϵ describes a first order displacement of P which can be understood as a section of its normal bundle; so the section s' of $\mathcal{L}$, restricted to P, can be interpreted as a section of the normal bundle to P in M.

[6]Since $c_i(P)$ is defined as a cohomology class of P, all classes appearing on the right hand side of these formulas should be restricted to P. We do not indicate this in the notation, and in any event will momentarily extend the classes away from P.

and

$$D = \frac{1}{6} \int_M P^3 \,, \tag{3.13}$$

we get

$$c_L = N_L^B = 6D + c_2 \cdot P \,, \tag{3.14}$$

$$c_R = N_R^B + \frac{1}{2} N_R^F = 6D + \frac{1}{2} c_2 \cdot P \,. \tag{3.15}$$

Here we have included in c_R also $2D + c_2 \cdot P/6$ complex fermions built out of $(0,2)$ forms as required by supersymmetry (there are no left moving fermions since $b_1 = 0$). Note that in comparing to section 2, D should be identified with the quantity of the same name introduced in equation (2.8). The p^A in (2.8) are the expansion coefficients of $[P]$ in a basis of classes Σ_A; thus $[P] = \sum_A p^A \Sigma_A$. In the basis Σ_A, the intersection form of M is given by $6D_{ABC} = \Sigma_A \cap \Sigma_B \cap \Sigma_C$.

The microscopic entropy for large q_0 is then from (3.4)

$$S_{micro} = 2\pi \sqrt{\frac{(6D + c_2 \cdot P) q_0}{6}} \,. \tag{3.16}$$

To compare with (2.9) we should expand in powers of $1/D$. One finds

$$S_{micro} = 2\pi \sqrt{D q_0} + c_2 \cdot P \frac{\pi}{6} \sqrt{\frac{q_0}{D}} + \ldots \tag{3.17}$$

This agrees with the Bekenstein-Hawking result (2.9) plus the one loop correction (2.15). It would be interesting to macroscopically reproduce the full series of corrections obtained microscopically in (3.16).

The computation given here for Calabi-Yau spaces can be repeated for K3 $\times T^2$ or T^6 with only minor differences. Again the generic fivebrane configuration is described by a smooth holomorphic map. At special points in the moduli space it degenerates into a set of fivebranes with D intersections. We can view the moduli as coming from "blow-up" modes associated to each intersection. In all cases we find that the sigma model is (0,4) if D is nonzero. For the $K3 \times T^2$ or T^6 case there are additional modes coming from b_1, which are subleading for large D.

3.2 Membrane charge

In this section we consider the effect of endowing our black hole with nonzero membrane charge q_A. Membrane charge is actually carried in the effective two-dimensional theory by the massless scalars that arise from dimensional reduction of the chiral two-form b; they were counted in section 3.1. The reason for this is that membrane charge is the flux of the self-dual three-form $h = db$ over cycles of the form $S^1 \times \Sigma$, with Σ a two-cycle in P. The fluxes for various Σ reduce in terms of the low energy physics on $R \times S^1$ to the winding numbers around S^1 of the various massless scalars that arise from reduction of b. It can be seen that all membrane charges arise in this fashion if P is in the Kahler cone.

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Thus, the membrane charge is a vector in the Narain lattice of the massless scalars. Now, actually there are two lattices of interest. The obvious lattice is $\Gamma = H^2(P, \mathbf{Z})$. It has a sublattice $\Gamma_M = H^2(M, \mathbf{Z})$, consisting of two-dimensional classes on P that can be extended over M.[7] The lattice Γ has signature (b_2^+, b_2^-), with $b_2^\pm$ computed in section 3.1. The lattice Γ_M has signature $(1, b_2(M) - 1)$. The last statement is proved as follows. Since M has holonomy $SU(3)$ (and not a proper subgroup of $SU(3)$), $H^{2,0}(M) = 0$. Hence $H^2(M, \mathbf{R})$ is generated by differential forms of type (1,1), and the sublattice Γ_M is entirely of type (1,1). By the Hodge index theorem, $H^{1,1}(P)$ has a self-dual part that is one-dimensional, generated by $[P]$ itself, so the positive signature part of Γ_M is at most one-dimensional. Since $[P]$ does extend over M, it is a vector in Γ_M, and hence Γ_M has signature $(1, b_2(M) - 1)$ with the positive signature subspace being generated by $[P]$.

At first sight it seems that, for a fivebrane wrapped on $S^1 \times P$, the membrane charge should be a vector in Γ. Actually, we should take it to be a vector in the sublattice Γ_M. There are two closely related reasons for this.

(1) From the point of view of the underlying M-theory on $R^4 \times S^1 \times M$, the conserved membrane charges are vectors in Γ_M, not in the larger lattice Γ. Thus, in discussing the macroscopic entropy in section 2, the membrane charge was a vector in Γ_M. We should expect that any fivebrane state, with a charge vector that initially is in Γ but not in Γ_M, can decay to a state with charge vector in Γ_M. (In this decay, the inner product of the charge vector with any vector in Γ_M will be conserved.)

(2) To build a BPS state, the membrane charge vector must be in Γ_M. In fact, the membrane charge vector must be integral, and the BPS condition requires that the membrane charge should be a sum of a left-moving vector and a multiple of P. This important statement will be justified at the end of the present section. Because Γ_M has signature $(1, b_2(M) - 1)$ with the plus part generated by $[P]$, a charge vector in Γ_M automatically obeys the necessary conditions for a BPS state. Generically, a vector not in Γ_M would not obey those conditions. The reason for the last assertion is that for P a very ample divisor in M, one has $H^{2,0}(P) \neq 0$,[8] which generically gives an obstruction to the existence of vectors not in Γ_M and obeying the desired conditions for a BPS state.

In the macroscopic discussion of section 2, the effect of such nonzero membrane charges on the macroscopic entropy is simply to shift q_0 in equation (2.9). From a microscopic point of view, this happens because membrane charge – in other words, a nonzero winding number of the scalars – shifts the ground state energy and momentum of the effective two-dimensional theory. This can be seen as follows.

We recall that the conformal field theory on $R \times S^1$ has one chiral boson, arising from reduction of the self-dual antisymmetric tensor b on the fivebrane, for every harmonic

[7] Γ_M is a sublattice of Γ because for P a very ample divisor in M, the restriction map from $H^2(M, \mathbf{Z})$ to $H^2(P, \mathbf{Z})$ is injective. This is so because an ample divisor P has a positive intersection with every divisor D in M; in fact, the triple intersection number $D \cap P \cap P$ is positive, being the volume of D in a Kahler metric whose Kahler class is P.

[8] In fact, by the Hodge index theorem, $b_2^+ = 1 + 2 \dim H^{2,0}(P)$. The formula for b_2^+ in section 3.1 thus gives a formula for $\dim H^{2,0}(P)$.

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two-form on P. Since we wish to consider the case of a membrane charge vector in Γ_M, we focus on contributions from two-forms α_A on P which arise as the restriction to P of closed two-forms α_A on M (we denote them with the same letter). For simplicity we assume that the Kahler class J of M is a multiple of P. This need not be the case in general, but the index of BPS states does not depend on the choice of J, so it suffices to consider this case. In any event, we have seen in section 2 that at the black hole horizon, J is a multiple of P.

We obtain two-dimensional fields ϕ^A by an ansatz $b = \sum_A \phi^A \alpha_A$ for the two-form potential. The fields ϕ^A are constrained by self-duality of the three-form field strength h in $R \times S^1 \times P$. In fact,

$$P^A_{\mp B} \partial_\pm \phi^B = 0\,, \tag{3.18}$$

where the projection operators $P_\pm$ are

$$P^A_{\pm B} = \tfrac{1}{2}(\delta^A{}_B \pm \frac{1}{6} D^{AC} g_{CB})\,, \tag{3.19}$$

with

$$g_{AB} = \int_P \alpha_A \wedge *\alpha_B = -6D_{AB} + \frac{12 D_{AC} p^C D_{BE} p^E}{D}\,, \tag{3.20}$$

and $*$ is the Hodge dual in P.

Let k^A be the winding numbers of the ϕ^A, and let $k^A_\pm = P^A_{\pm B} k^B$. For nonzero k there is a zero-mode contribution to the S^1 momentum given by

$$\frac{\Delta q_0}{R} = \int_{S^1} d\sigma (T_{--} - T_{++}) = 2\pi^2 R g_{AB} (k^A_+ k^B_+ - k^A_- k^B_-) = -12\pi^2 R D_{AB} k^A k^B\,. \tag{3.21}$$

We will now demonstrate more precisely that a state with nonzero momentum k^A has a nonzero flux of h and therefore caries membrane charge. The three-form potential $A^{(3)}$ of the low energy eleven-dimensional supergravity has couplings both to the world volume of a membrane, which we will take to be $R \times Q$, where R is parametrized by time and Q is a two-surface in space, and to the five-brane worldvolume $R \times S^1 \times P$. The couplings are

$$\int_{R\times Q} A^{(3)} + \int_{R\times S^1 \times P} h \wedge A^{(3)}\,. \tag{3.22}$$

The three form gauge potential gives rise to 3+1-dimensional $U(1)$ gauge fields A^A_μ, $\mu = 0, ..3$ via the decomposition

$$A^{(3)} = A^A_\mu dx^\mu \wedge \alpha_A\,. \tag{3.23}$$

Membrane charge acts as a source for these gauge fields. When k is nonzero, h becomes

$$h = \alpha_A k^A_+ \wedge dx^+ + \alpha_A k^A_- \wedge dx^-\,. \tag{3.24}$$

It follows from (3.22) and the relation

$$\int_P \alpha_A \wedge \alpha_B = 6D_{AB} \tag{3.25}$$

that the effect of nonzero h is to induce a membrane charge

$$q_A = 12\pi R D_{AB}(k_+^B - k_-^B)\,. \tag{3.26}$$

Hence one finds that the total *leftmoving* S^1 momentum is

$$q_0 = \hat{q}_0 - \frac{1}{12} D^{AB} q_A q_B\,, \tag{3.27}$$

where $\hat{q}_0$ is the momentum carried by non-zero modes of the $c_L \approx 6D$ conformal field theory and we have used $k_+^A k_-^B D_{AB} = 0$. Since the entropy counts the number of ways of distributing the momentum within these modes, the microscopic entropy is

$$S = 2\pi\sqrt{D\hat{q}_0}\,, \tag{3.28}$$

in full agreement with (2.9), including the coefficient and the sign.

We are interested in supersymmetric BPS states and so we still need to check that the momenta do not break supersymmetry. All BPS states have the same number of supersymmetries as the $k = 0$ ground state. Supersymmetry is realized by the right movers, so naively supersymmetry is broken whenever the right-moving momentum is nonzero. However, there is a crucial subtlety here. In fact, if the right-moving momentum is nonzero but is a multiple of $[P]$, then all the linearly realized supersymmetries of the vacuum are broken but an equal number of unbroken supersymmetries reappear as combinations of the original linear and nonlinear supersymmetries, as follows. The two-form b on the membrane world-volume has one distinguished right-moving mode coming from $b \sim [P]$. Let k_R be the momentum of this mode. This mode is paired together with the three translational zero modes and the four goldstinos ψ in a $(0,4)$ supermultiplet. For nonzero k_R (but no spacetime momentum), the ψ transform under the linear supersymmetries as

$$\delta\psi = k_R \gamma^4 \epsilon^{lin}\,, \tag{3.29}$$

signalling the breaking of the original supersymmetries. However as goldstinos they also transform under the nonlinear supersymmetries as

$$\delta\psi = \epsilon^{nlin}\,. \tag{3.30}$$

Hence linear combinations of transformations obeying

$$\epsilon^{nlin} = -k_R \gamma^4 \epsilon^{lin} \tag{3.31}$$

provide four unbroken supersymmetries for generic values of k_R, and BPS states exist for all k_R. Such mixing of linear and nonlinear supersymmetries in describing BPS states is of course familiar in many other aspects of brane physics, for instance in matrix theory.

The other right-moving scalars of the theory are not paired with the goldstinos in this way. So their momenta must vanish in a BPS state. Hence, the BPS condition asserts not that the right-moving charge is zero but that it is a multiple of $[P]$, as we asserted near the beginning of section 3.2 in explaining why the charge vector should lie in Γ_M.

4 α' corrections

In the preceding, we have considered corrections to the entropy which correspond to string loops in the type IIA picture. These come from higher dimension operators, and the leading correction is suppressed by a factor of $D^{-2/3}$. There are also α' corrections which arise at string tree level. These correct the prepotential which determines the leading low-energy action, and are suppressed by inverse powers of the string frame volume of M, $V_{str} = R^3 V_M$. Corrections arise at three-loops, four-loops and nonperturbatively in the string sigma model [22]. The leading three-loop correction to the entropy is supressed by a factor $D^{1/3}/q_0$. Consistency of the present analysis requires that this be smaller than the string loop suppression, or $q_0 \gg D$. This is the same as the condition $q_0 \gg c_L$ needed in the microscopic analysis.

It would be of interest to go beyond the present analysis and understand the leading α' correction, which amounts to an effective shift of $c_2 \cdot P/24$ to q_0. This was interpreted in the string theory picture in [8] as arising from the anomalous zerobrane charge of a fourbrane in curved space [23, 24]. In the M-theory picture it has the right form to arise from the ground state energy of the left-moving chiral bosons. Because there are $c_L = 6D + c_2 \cdot P$ such bosons, this is potentially given by $(6D + c_2 \cdot P)/24$ rather than $c_2 \cdot P/24$. In order to compute this shift one must know the boundary conditions. For reasons discussed in [25], these scalars can in general change by additive constants in going around S^1, but from [25] it follows that this effect alone can not shift q_0 appropriately. Indeed independent analyses using anomaly inflow [26, 27] have recently found an extra shift proportional to D. It would be interesting to understand this in the context of the semiclassical black hole entropy formula.

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A Normalization of the charge quantization conditions

Let us fix the normalization of the three form potential $A^{(3)}$ so that its coupling to the membrane is

$$\int_{\Sigma_3} A^{(3)} \,, \tag{A.1}$$

and normalize the self-dual field strength h so that its coupling on the fivebrane world-volume to the three form potential is

$$\int_{\Sigma_6} A^{(3)} \wedge h \,. \tag{A.2}$$

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The stress tensor on the fivebrane will contain a contribution from h

$$T_{\mu\nu} = B h_{\mu\rho\sigma} h_{\nu}^{\ \rho\sigma} \,, \tag{A.3}$$

where B is a constant to be determined. (Notice that $h_{\mu\rho\sigma}h^{\mu\rho\sigma} = 0$ due to the self-duality condition, so $T^{\mu}_{\mu} = 0$ as required for conformal invariance). In order to determine B, we consider the case of the torus and use the known BPS mass formula. Consider a bound state of a membrane and a fivebrane represented as an h flux in the fivebrane. The membrane is along 12 and the fivebrane along 12345. The mass of this bound state is

$$M = \sqrt{M_5^2 + M_2^2} \sim M_5 + \frac{1}{2}\frac{R_1 R_2}{(R_3 R_4 R_5)} \,, \tag{A.4}$$

where we used the relation $T_2^2/T_5 = 2\pi$ obtained in [28]. We write $A^{(3)} = A_\mu dx^\mu \beta_{12} dx^1 dx^2$ where $\int_{T_{12}} \beta_{12} = 1$. In this case the coupling (A.2) becomes

$$\int A_0 \; dt \; h_{345} (2\pi)^3 R_3 R_4 R_5 \,, \tag{A.5}$$

where the relation between forms and components is $h = \frac{1}{6} h_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho$. If h induces one unit of charge we should have

$$h_{345} = \frac{1}{(2\pi)^3 R_3 R_4 R_5} \,. \tag{A.6}$$

This can be seen by comparing (A.5) to (A.1). Inserting (A.6) in (A.3) and comparing it to (A.4) we find that

$$B = \frac{\pi}{2} \,. \tag{A.7}$$

Now we return to our case with a generic Calabi Yau and write

$$A^{(3)} = A_\mu^A dx^\mu \alpha_A \,, \tag{A.8}$$

where the integral of α_A over the corresponding cycle is normalized to one. Using the formula (3.24) for h and the coupling (3.22) with (A.8) we conclude that the membrane charge is (3.26), using $k_\pm^A \equiv \partial_\pm \phi^A$. Evaluating the momentum is straightforward from (A.3). Defining $x^\pm = t \pm x^{11}$ we have $T_{01} = T_{++} - T_{--}$. Using this relation and (3.24) we finally obtain (3.21).

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